Eur. Phys. J. D **12**, 11–13 (2000)

THE EUROPEAN PHYSICAL JOURNAL D
EDP Sciences

New formulation for squeezed fermion-pair states as a counterpart of Grassmannian evolution*

H.-y. Fan[1,2,3] and Z.-h. Sun[3,a]

[1] CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, P.R. China
[2] Department of Applied Physics, Shanghai Jiao Tong University, Shanghai 200030, P.R. China
[3] Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China

Received 17 February 2000

Abstract. A new formulation for deriving the squeezing fermion-pair operators, which manifestly shows the correspondence between pseudo-classical transformation in Grassmann number space and quantum mechanical Fermi operators, is presented. In this formulation new fermionic operator formulas regarding Bogoliubov-Valatin transformation are obtained.

PACS. 03.65.-w Quantum mechanics

In reference [1] it is pointed out that Bardeen-Cooper-Schreiffer state [2] is a sort of squeezed fermion-pair state. In a recent paper [3,4], Liu and Li proposed a formulation of deducing the Bogoliubov-Valatin transformation formula [5] by using the disentangling technique for matrices [6,7]. Using this formula they claimed that the BCS reduced Hamiltonian can be diagonalized in the particle space rather than the quasi-particle space. They started from the transformation which mixes the Fermi operators $a_{p\uparrow}$ and $a^\dagger_{-p\downarrow}$,

$$\begin{aligned} U_p a_{p\uparrow} U_p^\dagger &= \mu_p a_{p\uparrow} - \nu_p a^\dagger_{-p\downarrow}, \\ U_p a_{-p\downarrow} U_p^\dagger &= \mu_p a_{-p\downarrow} + \nu_p a^\dagger_{p\uparrow}, \\ U_p a^\dagger_{p\uparrow} U_p^\dagger &= \mu_p^* a^\dagger_{p\uparrow} - \nu_p^* a_{-p\downarrow}, \\ U_p a^\dagger_{-p\downarrow} U_p^\dagger &= \mu_p^* a^\dagger_{-p\downarrow} + \nu_p^* a_{p\uparrow}, \end{aligned} \tag{1}$$

where U_p is a unitary operator to be determined, μ_p and ν_p are complex transformation coefficients satisfying

$$|\mu_p|^2 + |\nu_p|^2 = 1, \tag{2}$$

then Liu and Li assigned that μ_p and ν_p are the functions of a certain real parameter x, tried to derive the unitary operator U_p by differentiating equation (1) with respect to x. They solved the resulting equation, with use of disentangling technique for matrices and finally led to equation (9) of [4]. However, their differential equations need tedious calculations and the knowledge of Lie algebra. In this Letter, we want to point out that U_p in particle space can be obtained very easily by the mapping technique from a pseudo-classical transformation

$$(\alpha_1, \alpha_2) \to \left(\mu_p^*\alpha_1 + \nu_p\bar{\alpha}_2, \mu_p^*\alpha_2 - \nu_p\bar{\alpha}_1\right)$$

in Grassmann number space onto the following ket-bra integration-form projection operator

$$U_p = \frac{1}{\mu_p^*}\int\int \mathrm{d}\bar{\alpha}_1\mathrm{d}\alpha_1\mathrm{d}\bar{\alpha}_2\mathrm{d}\alpha_2 \times \left|\mu_p^*\alpha_1 + \nu_p\bar{\alpha}_2, \mu_p^*\alpha_2 - \nu_p\bar{\alpha}_1\right\rangle \langle\alpha_1, \alpha_2|, \tag{3}$$

where $|\alpha_i\rangle$ $(i = 1, 2)$ is a normalized fermionic coherent state [8] defined as

$$\begin{aligned} |\alpha_1\rangle &= \exp[-\frac{1}{2}\bar{\alpha}_1\alpha_1 + a^\dagger_{p\uparrow}\alpha_1]\,|0\rangle_1, \\ |\alpha_2\rangle &= \exp[-\frac{1}{2}\bar{\alpha}_2\alpha_2 + a^\dagger_{-p\downarrow}\alpha_2]\,|0\rangle_2, \end{aligned} \tag{4}$$

α_i is the Grassmann number satisfying the anti-commutative relations

$$\{\alpha_1, \alpha_2\} = 0, \qquad \{\alpha_i, a_i\} = 0, \tag{5}$$

and the Berezin integration formula [11]

$$\int \mathrm{d}\alpha\,\alpha = 1, \qquad \int \mathrm{d}\alpha = 0. \tag{6}$$

The ket in equation (3) is constructed by the variation

$$\alpha_1 \to \mu_p^*\alpha_1 + \nu_p\bar{\alpha}_2, \qquad \alpha_2 \to \mu_p^*\alpha_2 - \nu_p\bar{\alpha}_1,$$

* This work is supported by the National Natural Science Foundation of China and Foundation of Educational Ministry of China.
[a] e-mail: zhsun@mail.ustc.edu.cn

$$U_p = \frac{1}{\mu_p^*}\int\int \mathrm{d}\bar{\alpha}_1\mathrm{d}\alpha_1\mathrm{d}\bar{\alpha}_2\mathrm{d}\alpha_2 \exp\Big\{-\frac{1}{2}\left[\left(\mu_p\bar{\alpha}_1+\nu_p^*\alpha_2\right)\left(\mu_p^*\alpha_1+\nu_p\bar{\alpha}_2\right)\right.$$
$$\left.+\ \left(\mu_p\bar{\alpha}_2-\nu_p^*\bar{\alpha}_1\right)\left(\mu_p^*\alpha_2-\nu_p\bar{\alpha}_1\right)\right] + a_{p\uparrow}^\dagger\left(\mu_p^*\alpha_1+\nu_p\bar{\alpha}_2\right) + a_{-p\downarrow}^\dagger\left(\mu_p^*\alpha_2-\nu_p\bar{\alpha}_1\right)\Big\}$$
$$\times \left|00\right\rangle\left\langle 00\right| \exp\left[-\frac{1}{2}\left(\bar{\alpha}_1\alpha_1+\bar{\alpha}_2\alpha_2\right)+\bar{\alpha}_1 a_{p\uparrow}+\bar{\alpha}_2 a_{-p\downarrow}\right]$$
$$= \frac{1}{\mu_p^*}\int\int \mathrm{d}\bar{\alpha}_1\mathrm{d}\alpha_1\mathrm{d}\bar{\alpha}_2\mathrm{d}\alpha_2 : \exp\Big\{-\left|\mu_p\right|^2\left(\bar{\alpha}_1\alpha_1+\bar{\alpha}_2\alpha_2\right)-\mu_p\nu_p\bar{\alpha}_1\bar{\alpha}_2-\mu_p^*\nu_p^*\alpha_2\alpha_1$$
$$+a_{p\uparrow}^\dagger\left(\mu_p^*\alpha_1+\nu_p\bar{\alpha}_2\right)+a_{-p\downarrow}^\dagger\left(\mu_p^*\alpha_2-\nu_p\bar{\alpha}_1\right)+\bar{\alpha}_1 a_{p\uparrow}+\bar{\alpha}_2 a_{-p\downarrow}-a_{p\uparrow}^\dagger a_{p\uparrow}-a_{-p\downarrow}^\dagger a_{-p\downarrow}\Big\} : \quad (8)$$

which is a pseudo-classical correspondence of (1). Once the integration in (3) is performed, the explicit form of U_p in Hilbert space can be obtained. At this point we would like to mention that Grassmann numbers and fermionic coherent states are recently employed to evaluate the nonadiabatic Hannay's angle of Grassmannian of spin one half in a varying magnetic field [9]. By substituting equation (4) into equation (3), taking into account the identity

$$\left|00\right\rangle\left\langle 00\right| =: \exp\left[-a_{p\uparrow}^\dagger a_{p\uparrow}-a_{-p\downarrow}^\dagger a_{-p\downarrow}\right] :, \quad (7)$$

where : : denotes normal ordering of Fermi operators, we have the following expression of U_p,

see equation (8) above.

In order to perform the integration, we use the technique of integration within an ordered product (IWOP) of Fermi operators [10] (note that Fermi operators are anti-permute within : :, but a Grassmann-number-Fermi-operator pair commutes with another such a pair within : :.) and employ the Grassmann number integral formula [11]

$$\int\prod_{i=1}\mathrm{d}\bar{\alpha}_i\mathrm{d}\alpha_i \times \exp\left[-\sum_{k,j}\bar{\alpha}_k A_{ij}\alpha_j+\sum_k\left(\bar{\alpha}_k\eta_k+\alpha_k\bar{\eta}_k\right)\right] = \det A\exp\left[\sum_{k,j}\bar{\eta}_k\left(A^{-1}\right)_{kj}\eta_j\right], \quad (9)$$

where $\bar{\eta}_k$ and η_k are both Grassmann numbers, A is a complex matrix. Then after the integration equation (8) becomes to

$$U_p = \mu_p\exp\left[\frac{\nu_p}{\mu_p}a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger\right] : \exp\left[\left(\frac{1}{\mu_p}-1\right)\times\left(a_{p\uparrow}^\dagger a_{p\uparrow}+a_{-p\downarrow}^\dagger a_{-p\downarrow}\right)\right] : \exp\left[\frac{\nu_p^*}{\mu_p}a_{p\uparrow}a_{-p\downarrow}\right]. \quad (10)$$

Using the Fermi operator identity,

$$\exp\left[\lambda f^\dagger f\right] = 1+\left(\mathrm{e}^\lambda-1\right)f^\dagger f =: \exp\left[\left(\mathrm{e}^\lambda-1\right)f^\dagger f\right] :, \quad (11)$$

where $f^\dagger$ (f) is Fermi creation (annihilation) operator, we can remove : : away in equation (10), this gives the final form of U_p, *i.e.*,

$$U_p = \exp\left[\frac{\nu_p}{\mu_p}a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger\right]\exp\left[-\left(a_{p\uparrow}^\dagger a_{p\uparrow}+a_{-p\downarrow}^\dagger a_{-p\downarrow}-1\right)\times\ln\mu_p\right]\exp\left[\frac{\nu_p^*}{\mu_p}a_{p\uparrow}a_{-p\downarrow}\right]. \quad (12)$$

From the operator identity

$$\mathrm{e}^A B\mathrm{e}^{-A} = B+[A,B]+\frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]]+\cdots, \quad (13)$$

one immediately checks that (12) leads to the unitary transform (1). It should be noticed that fermionic coherent states are not orthogonal each other, instead,

$$\left\langle\alpha\right|\left.\alpha'\right\rangle = \exp\left[-\frac{1}{2}\bar{\alpha}\alpha-\frac{1}{2}\bar{\alpha}'\alpha'+\bar{\alpha}\alpha'\right], \quad (14)$$

therefore U_p does not change a coherent state to another coherent state, *i.e.*,

$$U_p\left|\alpha_1',\alpha_2'\right\rangle \neq \left|\mu_p^*\alpha_1+\nu_p\bar{\alpha}_2,\mu_p^*\alpha_2-\nu_p\bar{\alpha}_1\right\rangle. \quad (15)$$

However, the following expectation values

$$\left\langle\alpha_1,\alpha_2\right|U_p a_{p\uparrow}U_p^\dagger\left|\alpha_1,\alpha_2\right\rangle = \mu_p\alpha_1-\nu_p\bar{\alpha}_2,$$
$$\left\langle\alpha_1,\alpha_2\right|U_p a_{-p\downarrow}U_p^\dagger\left|\alpha_1,\alpha_2\right\rangle = \mu_p\alpha_2+\nu_p\bar{\alpha}_1 \quad (16)$$

do indicate that the point (α_1,α_2) in pseudo-classical Grassmann number space has been moved to another point $(\mu_p\alpha_1-\nu_p\bar{\alpha}_2,\mu_p\alpha_2+\nu_p\bar{\alpha}_1)$. From above we see that the fermionic squeezing operator is just the mapping of its corresponding pseudo-classical transformation. The normally ordered squeezing fermion-pair operator can be easily derived by virtue of the fermion coherent state and the IWOP technique. This idea has more advantages. For example, due to (3) and (14) we can use the fermionic coherent state representation and (7) as well as (9) to derive

$$\exp\left[\frac{\mu_p^*\nu_p'+\nu_p\mu_p'}{\mu_p\mu_p'-\nu_p^*\nu_p'}a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger\right]\exp\left[-\left(a_{p\uparrow}^\dagger a_{p\uparrow}+a_{-p\downarrow}^\dagger a_{-p\downarrow}-1\right)\ln\left(\mu_p\mu_p'-\nu_p^*\nu_p'\right)\right]\exp\left[\frac{\mu_p\nu_p'^*+\nu_p^*\mu_p'^*}{\mu_p\mu_p'-\nu_p^*\nu_p'}a_{p\uparrow}a_{-p\downarrow}\right]=$$
$$\exp\left[\frac{\nu_p}{\mu_p}a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger\right]\exp\left[-\left(a_{p\uparrow}^\dagger a_{p\uparrow}+a_{-p\downarrow}^\dagger a_{-p\downarrow}-1\right)\ln\mu_p\right]\exp\left[\frac{\nu_p^*}{\mu_p}a_{p\uparrow}a_{-p\downarrow}\right]$$
$$\times\exp\left[\frac{\nu_p'}{\mu_p'}a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger\right]\exp\left[-\left(a_{p\uparrow}^\dagger a_{p\uparrow}+a_{-p\downarrow}^\dagger a_{-p\downarrow}-1\right)\ln\mu_p'\right]\exp\left[\frac{\nu_p'^*}{\mu_p'}a_{p\uparrow}a_{-p\downarrow}\right] \quad (21)$$

the result of two successive fermionic squeezing transformations,

$$U_pU_p'=\frac{1}{\mu_p^*\mu_p'^*}\int\int\int\int \mathrm{d}\bar\alpha_1\mathrm{d}\alpha_1\mathrm{d}\bar\alpha_2\mathrm{d}\alpha_2\mathrm{d}\bar\alpha_1'\mathrm{d}\alpha_1'\mathrm{d}\bar\alpha_2'\mathrm{d}\alpha_2'$$
$$\times\left|\mu_p^*\alpha_1+\nu_p\bar\alpha_2,\mu_p^*\alpha_2-\nu_p\bar\alpha_1\right\rangle$$
$$\times\left\langle\alpha_1,\alpha_2\right|\mu_p'^*\alpha_1'+\nu_p'\bar\alpha_2',\mu_p'^*\alpha_2'-\nu_p'\bar\alpha_1'\rangle\left\langle\alpha_1',\alpha_2'\right|. \quad (17)$$

Note that U_p and U_p' do not commute each other. After performing the integration, we get its explicit expression,

$$U_pU_p'=\exp\left[\frac{\mu_p^*\nu_p'+\nu_p\mu_p'}{\mu_p\mu_p'-\nu_p^*\nu_p'}a_{p\uparrow}^\dagger a_{-p\downarrow}^\dagger\right]$$
$$\times\exp\left[-\left(a_{p\uparrow}^\dagger a_{p\uparrow}+a_{-p\downarrow}^\dagger a_{-p\downarrow}-1\right)\ln\left(\mu_p\mu_p'-\nu_p^*\nu_p'\right)\right]$$
$$\times\exp\left[\frac{\mu_p\nu_p'^*+\nu_p^*\mu_p'^*}{\mu_p\mu_p'-\nu_p^*\nu_p'}a_{p\uparrow}a_{-p\downarrow}\right]. \quad (18)$$

To show the correctness of (18), we calculate

$$U_pU_p'a_{p\uparrow}U_p'^\dagger U_p^\dagger=\left(\mu_p\mu_p'-\nu_p^*\nu_p'\right)a_{p\uparrow}-\left(\mu_p^*\nu_p'+\nu_p\mu_p'\right)a_{-p\downarrow}^\dagger. \quad (19)$$

It is noted that

$$\left|\mu_p\mu_p'-\nu_p^*\nu_p'\right|^2+\left|\mu_p^*\nu_p'+\nu_p\mu_p'\right|^2=1,$$

since $|\mu_p|^2+|\nu_p|^2=1$, and$|\mu_p'|^2+|\nu_p'|^2=1$. On the other hand, according to equation (1) we have

$$U_p\left(U_p'a_{p\uparrow}U_p'^\dagger\right)U_p^\dagger=U_p\left(\mu_p'a_{p\uparrow}-\nu_p'a_{-p\downarrow}^\dagger\right)U_p^\dagger$$
$$=\mu_p'\left(\mu_pa_{p\uparrow}-\nu_pa_{-p\downarrow}^\dagger\right)-\nu_p'\left(\mu_p^*a_{-p\downarrow}^\dagger+\nu_p^*a_{p\uparrow}\right)$$
$$=\left(\mu_p\mu_p'-\nu_p^*\nu_p'\right)a_{p\uparrow}-\left(\mu_p^*\nu_p'+\nu_p\mu_p'\right)a_{-p\downarrow}^\dagger, \quad (20)$$

which is equivalent to equation (20). Thus the accordance of equation (21) with (20) demonstrates the correctness of (18). Combining equations (12, 18) together we obtain a new operator identity

see equation (21) above.

Following the same procedures as deriving (17) it is interesting to see that after the two successive transformation the point (α_1,α_2) in pseudo-classical Grassmann number space moves to the point

$$((\mu_p\mu_p'-\nu_p^*\nu_p')\alpha_1-(\mu_p^*\nu_p'+\nu_p\mu_p')\bar\alpha_2,$$
$$(\mu_p\mu_p'-\nu_p^*\nu_p')\alpha_2+(\mu_p^*\nu_p'+\nu_p\mu_p')\bar\alpha_1).$$

The formulation in this work shows the Grassmannian evolution is the classical counterpart of squeezing fermion-pair transformation. This may be of the same importance with the recent paper [9] in which the spin one half in a varying magnetic field is associated with the evolution of Grassmannian invariant-variable-angle coherent state.

References

1. K. Svozil, Phys. Rev. Lett. **65**, 3341 (1990).
2. J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
3. W.S. Liu, X.P. Li, Eur. Phys. J. D **2**, 1 (1998).
4. W.S. Liu, X.P. Li, Eur. Phys. J. D **6**, 149 (1999).
5. N.N. Bogoliubov, J. Exp. Theor. Phys. USSR **34**, 58 (1958); D.G. Valatin, Nuovo Cimento **7**, 843 (1958).
6. R.A. Fisher, M.M. Nieto, V.D. Sandberg, Phys. Rev. D **29**, 1107 (1984).
7. R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (John Wiley & Sons, New York, 1974).
8. Y. Ohnuki, T. Kashiwa, Prog. Theor. Phys. **60**, 548 (1978).
9. M. Maamache, O. Cherbal, Eur. Phys. J. D **6**, 145 (1999).
10. H.Y. Fan, Phys. Rev. A **40**, 4237 (1990).
11. F.A. Berezin, *The Method of Second Quantization* (New York, Academic Press, 1966).